**Logarithmic Inequalities**

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If $a$ and $b$ are distinct positive real numbers, prove that

$$\sqrt{ab} < \frac{a-b}{\ln a - \ln b} < \frac{a+b}{2}$$

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Due to symmetry of the inequality we may assume that $a > b$. Let $x := \ln \frac{a}{b}$.

Then $x > 0$ and $\sqrt{ab} < \frac{a-b}{\ln a - \ln b} < \frac{a+b}{2} \iff \sqrt{\frac{a}{b}} < \frac{\frac{a}{b}-1}{\ln \frac{a}{b}} < \frac{\frac{a}{b}+1}{2} \iff$

(**1**) $\quad e^{x/2} < \frac{e^x-1}{x} < \frac{e^x+1}{2}.$

First we will prove that $e^{x/2} < \frac{e^x-1}{x}$, for any $x > 0$.

Since $e^{x/2} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{x^n}{2^n n!} \iff 1 + xe^{x/2} = 1 + x + \frac{x^2}{2} + \sum_{n=2}^{\infty} \frac{x^{n+1}}{2^n n!} =$

$1 + x + \frac{x^2}{2} + \sum_{n=3}^{\infty} \frac{x^n}{2^{n-1}(n-1)!}$ and $e^x = 1 + x + \frac{x^2}{2} + \sum_{n=3}^{\infty} \frac{x^n}{n!}$ then

$e^x - (1 + xe^{x/2}) = \sum_{n=3}^{\infty} x^n \left( \frac{1}{n!} - \frac{1}{2^{n-1}(n-1)!} \right) > 0$ because $\frac{1}{n!} - \frac{1}{2^{n-1}(n-1)!} =$

$\frac{1}{2^{n-1} n!}(2^{n-1} - n)$ and $2^{n-1} > n$ for any $n \geq 3$ (by Math Induction).

Hence $1 + xe^{x/2} < e^x \iff e^{x/2} < \frac{e^x-1}{x}$.

Also we have $\frac{e^x-1}{x} < \frac{e^x+1}{2} \iff 2e^x - 2 < xe^x + x \iff e^x < 1 + \frac{x}{2} + \frac{1}{2}xe^x \iff$

$1 + x + \sum_{n=2}^{\infty} \frac{x^n}{n!} < 1 + \frac{x}{2} + \frac{1}{2}x\left(1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}\right) \iff \sum_{n=2}^{\infty} \frac{x^n}{n!} < \sum_{n=1}^{\infty} \frac{x^{n+1}}{2 \cdot n!} \iff$

$\sum_{n=2}^{\infty} \frac{x^n}{n!} < \sum_{n=2}^{\infty} \frac{x^n}{2 \cdot (n-1)!}$ where latter inequality holds for any $x > 0$ since

$\frac{1}{n!} \leq \frac{1}{2 \cdot (n-1)!} \iff 2 \leq n.$